resonant frequency is increased as s increased. This behavior is quite different from that for the case of $\varepsilon_1 = \varepsilon_2 = 2.32$, where the resonant frequency is decreased as s is increased. This is probably due to the effective permittivity of the region under the patch being lowered with the existence of the airgap. As for the imaginary resonant frequencies, the results are shown in Fig. 2(b). It is seen that with an airgap the imaginary resonant frequency is higher, i.e., the radiation loss of the structure is increased. The spherical structure is also seen to be a more efficient radiator than the planar structure. Fig. 3 shows the half-power bandwidth of the microstrip structure for the case in Fig. 2. The bandwidth is seen to be considerably increased due to the existence of an airgap and the spherical structure is also with a higher bandwidth as compared to the planar structure.

IV. Conclusions

The geometry of the spherical-circular microstrip structure with an airgap is studied. Complex resonant frequencies at TM_{11} mode for both the spherical and planar structures are presented. Results indicate that the radiation loss of the microstrip structure increases as the airgap thickness is increased and the spherical structure with an airgap is also a more efficient radiator than the planar structure with an airgap. Furthermore, the half-power bandwidth of the microstrip structure is considerably increased due to the existence of an airgap, and the bandwidth of the spherical structure is also greater than that of the planar structure. This improves the low bandwidth characteristics of the microstrip structure.

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Useful Bessel Function Identities and Integrals

E. B. Manring and J. Asmussen, Jr.

***Abstract*—A number of previously unpublished Bessel function identities and indefinite integrals are listed. These are useful in solving electromagnetics problems in cylindrical coordinates, including energy and power calculations, and mode orthogonalization in lossy media. Using these integrals in conjunction with two previously published indefinite Bessel function integrals, two orthogonality integrals are derived. Values of the indefinite integrals at limits of zero and infinity are also given.**

I. Introduction

Often in the solution of electromagnetics problems in cylindrical coordinates, products of Bessel functions are encountered in indefinite integrals or elsewhere. Examples are orthogonalization integrals of modal expansions for matching fields across discontinuities, calculation of power dissipated and energy stored in cylindrical waveguides and cavities, or similar calculations for cylindrical dielectric waveguides [1]. While there are tabulated solutions to certain of these integrals [2], [3], some cannot be found in common references.

The integrals given in [1] are needed in energy and power calculations in geometries of circular symmetry with lossless materials. However, in lossy media, the radial wavenumbers, k_ρ, are complex. In that case $k_\rho \neq k_\rho^*$ and the integrals in [1] cannot be used. Two new indefinite integrals are given below which account for the $k_\rho \neq k_\rho^*$ case. These integrals are also needed when using one mode to orthogonalize a modal series expansion of electric or magnetic fields in a waveguide or cavity at an axial junction. In addition to these integrals, three recurrence identities and another indefinite integral are listed which may be useful in other instances. Since indefinite integrals are often evaluated at limits of zero and infinity, the values of the integrals at these limits are enumerated here. Employing the limiting case at zero for the ordinary Bessel function integral in conjunction with one of the integral identities in [1], certain orthogonality properties of the ordinary Bessel functions may be derived. Two of them, which appear in orthogonality integrals for homogeneously loaded waveguides, are given below.

II. Indefinite Integrals

Given F_ν and G_ν, such that

$$\begin{aligned} F_\nu(\alpha z) &= AJ_\nu(\alpha z) + BY_\nu(\alpha z), \\ F_\nu'(\alpha z) &= AJ_\nu'(\alpha z) + BY_\nu'(\alpha z) \\ G_\nu(\beta z) &= CJ_\nu(\beta z) + DY_\nu(\beta z), \\ G_\nu'(\beta z) &= CJ_\nu'(\beta z) + DY_\nu'(\beta z) \end{aligned} \tag{12}$$

where J_ν and Y_ν are ordinary Bessel functions of the first and second kinds with complex arguments, and ν is an arbitrary complex constant, it is possible to show that

$$\begin{aligned} \int \left[F_\nu'(\alpha z)G_\nu'(\beta z) + \frac{\nu^2}{\alpha\beta z^2} F_\nu(\alpha z)G_\nu(\beta z)\right] z\, dz \\ = \frac{z}{\alpha^2 - \beta^2}\left[\alpha F_\nu(\alpha z)G_\nu'(\beta z) - \beta F_\nu'(\alpha z)G_\nu(\beta z)\right], \\ \alpha \neq \beta. \end{aligned} \tag{13}$$

Since the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are linear combinations of J_ν and Y_ν, (2) is also true if F_ν and G_ν contain linear combinations

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of Hankel functions. The integral may be checked by differentiating the right-hand side and showing that it is equal to the integrand.

In order to perform the differentiation, it is necessary to have an expression for the second derivatives of F_ν and G_ν. These are found by examining the differential equation defining ordinary Bessel functions

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + \left(z^2 - \nu^2\right) w = 0. \tag{14}$$

The same integral relationship as is contained in (2) may be derived for modified Bessel functions. Given L_ν and M_ν, such that

$$\begin{aligned} L_\nu(\alpha z) &= P I_\nu(\alpha z) + Q K_\nu(\alpha z), \\ L'_\nu(\alpha z) &= P I'_\nu(\alpha z) + Q K'_\nu(\alpha z) \\ M_\nu(\beta z) &= R I_\nu(\beta z) + S K_\nu(\beta z), \\ M'_\nu(\beta z) &= R I'_\nu(\beta z) + S K'_\nu(\beta z), \end{aligned} \tag{15}$$

where I_ν and K_ν are modified Bessel functions, it may be shown that

$$\begin{aligned} \int \left[L'_\nu(\alpha z) M'_\nu(\beta z) + \frac{\nu^2}{\alpha\beta z^2} L_\nu(\alpha z) M_\nu(\beta z) \right] z\, dz \\ = \frac{z}{\alpha^2 - \beta^2} \left[\alpha L_\nu(\alpha z) M'_\nu(\beta z) - \beta L'_\nu(\alpha z) M_\nu(\beta z) \right]. \\ \alpha \neq \beta. \end{aligned} \tag{16}$$

Equation (5) is true for K_ν and $e^{i\pi\nu} K_\nu$ formulations of the modified Bessel function of the second kind, using the notation of [4]. This integral is identical to the one given above for ordinary Bessel functions, and can be checked by differentiation using the general differential equation for the modified Bessel functions

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - \left(z^2 + \nu^2\right) w = 0. \tag{17}$$

When ordinary and modified Bessel functions are combined in the form found in the integrals of (2) and (5), the integral is different. Using the previously defined functions F_ν and M_ν, the following integral relationship may be established:

$$\begin{aligned} \int \left[F'_\nu(\alpha z) M'_\nu(\beta z) + \frac{\nu^2}{\alpha\beta z^2} F_\nu(\alpha z) M_\nu(\beta z) \right] z\, dz \\ = \frac{z}{\alpha^2 + \beta^2} \left[\alpha F_\nu(\alpha z) M'_\nu(\beta z) + \beta F'_\nu(\alpha z) M_\nu(\beta z) \right]. \end{aligned} \tag{18}$$

This relationship is true for linear combinations of the modified Bessel functions I_ν, K_ν, and $e^{i\pi\nu} K_\nu$, and for the ordinary Bessel functions including Hankel functions.

The integrals of (2) and (5) are valid for $F_\nu = G_\nu$ and $L_\nu = M_\nu$, but in general it is necessary that α and β be different. For the case where $\alpha = \beta$, a solution does exist when the two combining functions are identical. Integrals for the case $\alpha = \beta$ with $F_\nu = G_\nu$ for ordinary Bessel functions, and $\alpha = \beta$ with $L_\nu = M_\nu$ for modified Bessel functions, are given by [1] and repeated here:

$$\begin{aligned} \int \left[F'^2_\nu(\alpha z) + \frac{\nu^2}{\alpha^2 z^2} F^2_\nu(\alpha z) \right] z\, dz \\ = \frac{z^2}{2} \left[F'^2_\nu(\alpha z) + F^2_\nu(\alpha z)\left(1 - \frac{\nu^2}{\alpha^2 z^2}\right) \right. \\ \left. + \frac{2}{\alpha z} F_\nu(\alpha z) F'_\nu(\alpha z) \right] \end{aligned} \tag{19}$$

$$\begin{aligned} \int \left[L'^2_\nu(\alpha z) + \frac{\nu^2}{\alpha^2 z^2} L^2_\nu(\alpha z) \right] z\, dz \\ = \frac{z^2}{2} \left[L'^2_\nu(\alpha z) - L^2_\nu(\alpha z)\left(1 + \frac{\nu^2}{\alpha^2 z^2}\right) \right. \\ \left. + \frac{2}{\alpha z} L_\nu(\alpha z) L'_\nu(\alpha z) \right]. \end{aligned} \tag{20}$$

III. Recurrence Identities

The above integrals, as noted, may be checked by differentiation. However, derivation of them proceeds in a straightforward manner from the following identities:

$$\begin{aligned} F'_\nu(\alpha z) G'_\nu(\beta z) \pm \frac{\nu^2}{\alpha\beta z^2} F_\nu(\alpha z) G_\nu(\beta z) \\ = \pm \frac{1}{2} \left[F_{\nu-1}(\alpha z) G_{\nu\mp 1}(\beta z) \right. \\ \left. + F_{\nu+1}(\alpha z) G_{\nu\pm 1}(\beta z) \right] \end{aligned} \tag{21}$$

$$\begin{aligned} L'_\nu(\alpha z) M'_\nu(\beta z) \pm \frac{\nu^2}{\alpha\beta z^2} L_\nu(\alpha z) M_\nu(\beta z) \\ = \frac{1}{2} \left[L_{\nu-1}(\alpha z) M_{\nu\mp 1}(\beta z) \right. \\ \left. + L_{\nu+1}(\alpha z) M_{\nu\pm 1}(\beta z) \right] \end{aligned} \tag{22}$$

$$\begin{aligned} F'_\nu(\alpha z) M'_\nu(\beta z) \pm \frac{\nu^2}{\alpha\beta z^2} F_\nu(\alpha z) M_\nu(\beta z) \\ = \frac{1}{2} \left[F_{\nu-1}(\alpha z) M_{\nu\mp 1}(\beta z) \right. \\ \left. - F_{\nu+1}(\alpha z) M_{\nu\pm 1}(\beta z) \right]. \end{aligned} \tag{23}$$

These identities may be derived from the recurrence relationships for ordinary and modified Bessel functions [5]. They are true only when the coefficients A, B, C, D, P, Q, R, and S are independent of ν, and Q and S are replaced by $e^{i\pi\nu} Q$ and $e^{i\pi\nu} S$. However, the integrals given above are not dependent upon such conditions as can be shown by the verification by differentiation.

wn by the verification by differentiation.

Using integral 5.54.1 of [3] (note: the 6th printing of the 1980 edition contains an error which is not present in the 1st printing; the "+" sign in the numerator should be a "−") or 11.3.29 of [2], with the identities of (10) and (11) above, the integrals of (2) and (5) may be found. The integral of (7) was derived by trial-and-error differentiation.

IV. Evaluation of the Integral Expressions at Zero and Infinity

The indefinite integrals given above must often be evaluated at limits of 0 and ∞. The results vary depending upon the Bessel functions included in the linear combinations which make up F_ν, G_ν, L_ν, and M_ν. The simplest means of presenting the values of the integral expressions at these limits is in terms of the coefficients A, B, C, D, P, Q, R, and S.

A. Evaluation at $z = 0$

At $z = 0$, the right-hand side of the integral of (2)—call it I1—for $\nu = 0$ is given by

$$\lim_{z\to 0} \mathrm{I1}\Big|_{\nu=0} = \begin{cases} \lim_{z\to 0} \frac{BD}{\alpha\beta} \ln(z) & B, D \neq 0 \\ -\frac{2\beta}{\pi\alpha} \frac{BC}{(\alpha^2-\beta^2)} & D = 0 \\ 0 & B, D = 0 \end{cases} \tag{24}$$

and, in general ($\mathrm{Re}\, \nu > 0$),

$$\lim_{z\to 0} \mathrm{I1} = \begin{cases} \lim_{z\to 0} \frac{-BD(\nu!)^2}{\nu(\alpha\beta)^{\nu+1}\left(\frac{z}{2}\right)^{2\nu}} & B, D \neq 0 \\ -\frac{(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)} \frac{BC\beta^{\nu-1}}{\pi\alpha^{\nu+1}} & D = 0 \\ 0 & B, D = 0. \end{cases} \tag{25}$$

For the integral of (5)—call it I2— the limit at $z = 0$ with $\nu = 0$ is

$$\lim_{z\to 0} \mathrm{I2}\Big|_{\nu=0} = \begin{cases} \lim\limits_{z\to 0} \frac{QS}{\alpha\beta}\ln(z) & Q, S \neq 0 \\ \frac{\beta}{\alpha}\frac{QR}{(\alpha^2-\beta^2)} & S = 0 \\ 0 & S, Q = 0 \end{cases} \tag{26}$$

and, in general ($\mathrm{Re}\ \nu > 0$),

$$\lim_{z\to 0} \mathrm{I2} = \begin{cases} \lim\limits_{z\to 0} \frac{-QS(\nu!)^2}{\nu(\alpha\beta)^{\nu+1}\left(\frac{z}{2}\right)^{2\nu}} & Q, S \neq 0 \\ \frac{(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)}\frac{QR\beta^{\nu-1}}{2\alpha^{\nu+1}} & S = 0 \\ 0 & Q, S = 0. \end{cases} \tag{27}$$

The limit at $z = 0$ for the integral of (7)—call it I3—for $\nu = 0$ is

$$\lim_{z\to 0} \mathrm{I3}\Big|_{\nu=0} = \begin{cases} \lim\limits_{z\to 0} \frac{-BS}{\alpha\beta}\ln(z) & B, S \neq 0 \\ \frac{-2\beta}{\alpha\pi}\frac{BR}{(\alpha^2-\beta^2)} & S = 0 \\ \frac{-\alpha}{\beta}\frac{AS}{(\alpha^2-\beta^2)} & B = 0 \\ 0 & B, S = 0, \end{cases} \tag{28}$$

and, in general ($\mathrm{Re}\ \nu > 0$),

$$\lim_{z\to 0} \mathrm{I3} = \begin{cases} \lim\limits_{z\to 0} \frac{BS(\nu!)^2}{\nu(\alpha\beta)^{\nu+1}\left(\frac{z}{2}\right)^{2\nu}} & B, S \neq 0 \\ -\frac{(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)}\frac{BR\beta^{\nu-1}}{\pi\alpha^{\nu+1}} & S = 0 \\ -\frac{(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)}\frac{AS\alpha^{\nu-1}}{2\beta^{\nu+1}} & B = 0 \\ 0 & B, S = 0. \end{cases} \tag{29}$$

The limit at $z = 0$ for the integral of (8)—call it I4—for $\nu = 0$ is

$$\lim_{z\to 0} \mathrm{I4}\Big|_{\nu=0} = \begin{cases} \lim\limits_{z\to 0} \frac{4B^2}{\pi^2\alpha^2}\ln(z) & B \neq 0 \\ 0 & B = 0 \end{cases} \tag{30}$$

and, in general ($\mathrm{Re}\ \nu > 0$),

$$\lim_{z\to 0} \mathrm{I4} = \begin{cases} \lim\limits_{z\to 0} B^2\left(\frac{\nu!}{\pi\alpha}\right)^2 \frac{\left(\frac{1}{2}-\frac{1}{\nu}\right)}{\left(\frac{1}{2}\alpha z\right)^{2\nu}} & B \neq 0 \\ 0 & B = 0. \end{cases} \tag{31}$$

The limit at $z = 0$ for the integral of (9)—call it I5—for $\nu = 0$ is

$$\lim_{z\to 0} \mathrm{I5}\Big|_{\nu=0} = \begin{cases} \lim\limits_{z\to 0} \frac{Q^2}{\alpha^2}\ln(z) & Q \neq 0 \\ 0 & Q = 0 \end{cases} \tag{32}$$

and, in general ($\mathrm{Re}\ \nu > 0$),

$$\lim_{z\to 0} \mathrm{I5} = \begin{cases} \lim\limits_{z\to 0} Q^2\left(\frac{\nu!}{\alpha}\right)^2 \frac{\left(1+\frac{3}{4\nu}\right)}{\left(\frac{1}{2}\alpha z\right)^{2\nu}} & Q \neq 0 \\ 0 & Q = 0. \end{cases} \tag{33}$$

If a limit at zero is encountered with $\mathrm{Re}\ \nu < 0$, the ordinary Bessel functions may be transformed by

$$F_{-\nu}(\alpha z) = [A\cos(\nu\pi) + B\sin(\nu\pi)]J_\nu(\alpha z) + [-A\sin(\nu\pi) + B\cos(\nu\pi)]Y_\nu(\alpha z). \tag{34}$$

The modified Bessel functions may be transformed such that

$$L_{-\nu}(\alpha z) = PI_\nu(\alpha z) + \left[Q + P\frac{2}{\pi}\sin(\nu\pi)\right]K_\nu(\alpha z). \tag{35}$$

B. Evaluation at $z = \infty$

For the limits at $z = \infty$, there are no special case limits for $\nu = 0$. However, there are many possible results depending on the values of α, β, and the coefficients A, B, C, D, P, Q, R, and S. Due to the large number of possible results, the expressions are left in simplest general limit form.

For the integral of (2), the limit at ∞ is given by

$$\lim_{z\to\infty} \mathrm{I1} = \lim_{z\to\infty} \frac{2}{\pi\sqrt{\alpha\beta}(\alpha^2-\beta^2)}\Big\{\alpha(A\cos\chi_\alpha + B\sin\chi_\alpha)(-C\sin\chi_\beta + D\cos\chi_\beta) - \beta(-A\sin\chi_\alpha + B\cos\chi_\alpha)(C\cos\chi_\beta + D\sin\chi_\beta)\Big\} \tag{36}$$

where $\chi_\alpha = \alpha z - (1/2\nu + 1/4)\pi$ and $\chi_\beta = \beta z - (1/2\nu + 1/4)\pi$.

The limit at ∞ of the integral of (5) is

$$\lim_{z\to\infty} \mathrm{I2} = \lim_{z\to\infty} \frac{1}{2\sqrt{\alpha\beta}}\left\{\frac{1}{\alpha+\beta}\left[\frac{1}{\pi}PRe^{z(\alpha+\beta)} - \pi QSe^{-z(\alpha+\beta)}\right] + \frac{1}{\alpha-\beta}\left[QRe^{-z(\alpha-\beta)} - PSe^{z(\alpha-\beta)}\right]\right\}. \tag{37}$$

The limit at ∞ for the integral of (7) is

$$\lim_{z\to\infty} \mathrm{I3} = \lim_{z\to\infty} \frac{1}{\sqrt{\pi\alpha\beta}(\alpha^2+\beta^2)} \cdot \Big\{e^{z\beta}R\big[A(\alpha\cos\chi_\alpha - \beta\sin\chi_\alpha) + B(\alpha\sin\chi_\alpha + \beta\cos\chi_\alpha)\big] - \pi e^{-z\beta}S\big[A(\alpha\cos\chi_\alpha + \beta\sin\chi_\alpha) + B(\alpha\sin\chi_\alpha - \beta\cos\chi_\alpha)\big]\Big\}. \tag{38}$$

The limits at ∞ for the integrals of (8) and (9) are given by

$$\lim_{z\to\infty} \mathrm{I4} = \lim_{z\to\infty}\Big\{\frac{z}{\pi\alpha}(A^2+B^2) + \frac{2}{\pi\alpha^2} \cdot \big[\sin\chi_\alpha\cos_{\chi_\alpha}(B^2-A^2) + (1-2\sin^2\chi_\alpha)AB\big]\Big\}. \tag{39}$$

$$\lim_{z\to\infty} \mathrm{I5} = \lim_{z\to\infty}\left[P^2\frac{e^{2\alpha z}}{2\pi\alpha^2} + Q^2\frac{\pi e^{-2\alpha z}}{2\alpha^2} - PQ\frac{z}{2\alpha}\right]. \tag{40}$$

C. Orthogonality Integrals

The integrals of (2) and (8), in conjunction with the limiting values given by (19) and (20), may be used to prove the following two orthogonality relationships for ordinary Bessel functions of the first kind:

$$\int_0^b \left[J_n'\left(\frac{\lambda_{np}}{b}\rho\right)J_n'\left(\frac{\lambda_{nq}}{b}\rho\right) + n^2\frac{b^2}{\lambda_{np}\lambda_{nq}\rho^2} \cdot J_n\left(\frac{\lambda_{np}}{b}\rho\right)J_n\left(\frac{\lambda_{nq}}{b}\rho\right)\right]\rho\,d\rho = \begin{cases} 0 & p \neq q \\ \frac{1}{2}[bJ_n'(\lambda_{np})]^2 & p = q \end{cases} \tag{41}$$

$$\int_0^b \left[J_n'\left(\frac{\lambda_{np}'}{b}\rho\right)J_n'\left(\frac{\lambda_{nq}'}{b}\rho\right) + n^2\frac{b^2}{\lambda_{np}'\lambda_{nq}'\rho^2}\right.$$

$$\cdot J_n\left(\frac{\lambda'_{np}}{b}\rho\right)J_n\left(\frac{\lambda'_{nq}}{b}\rho\right)\Bigg]\rho\,d\rho$$

$$= \begin{cases} 0 & p \neq q \\ \frac{b^2}{2}\left(1-\frac{n^2}{\lambda'^2_{np}}\right)J_n^2(\lambda'_{np}) & p = q \end{cases} \quad (42)$$

where λ_{np} and λ'_{np} are the zeros of ordinary Bessel functions and their derivatives, respectively. These integrals appear, respectively, in orthogonality relationships between TM and TE modes in homogeneously filled waveguides, i.e.,

$$\iint E^{TM}_{t_i} \cdot E^{TM}_{t_j}\, ds = 0 \qquad i \neq j \quad (43)$$

and

$$\iint E^{TE}_{t_i} \cdot E^{TE}_{t_i}\, ds = 0 \qquad i \neq j\,. \quad (44)$$

V. Summary

Indefinite Bessel function integrals useful in solving electromagnetics problems in lossy media with circular symmetric geometry have been presented for ordinary Bessel functions of the first and second kinds, for modified Bessel functions, and for combinations of ordinary and modified Bessel functions. Two orthogonal definite integrals have been presented for Bessel functions of the first kind. Additionally, six recurrence identities for similar combinations of Bessel functions have been presented. Limiting values of the integrals at zero and infinity have been given to facilitate their use in practical application. These integrals and recurrence identities are useful in any analysis that deals with products of entities which are solutions to the Helmholtz equation in cylindrical coordinates.

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Theory for a Cylindrical Pillbox Accelerator Cavity Using Layered Structures for Reducing Skin-Effect Losses

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Abstract—**It is shown that for a cylindrical pillbox accelerator cavity operating in a TM_{0n0} mode, the use of laminated conductors for the flat walls in conjunction with a multilayered dielectric structure for the round walls can decrease skin-effect losses by an order of magnitude over that of a copper cavity having the same accelerating field. The layered dielectric structure for the round walls works in a fashion similar to a quarter-wave interferometer. The laminated conductor on the flat walls reduces the ohmic losses by effectively increasing the skin depth.**

I. Introduction

Accelerator designs for nuclear and particle physics and free-electron laser applications have, in general, been either room-temperature copper or superconducting. Superconducting accelerators, besides having problems such as higher order mode dissipation, require the use of exotic fabrication techniques and the high cost and complexity of cryogenic systems. Acceleration gradients or duty factors in room temperature systems may be limited by the power loss, P_{loss}. A fair measure of the importance of wall losses in different cavity designs is the shunt impedance

$$R_s = \frac{\left(\int E_z\, dz\right)^2}{P_{\text{loss}}} \quad (45)$$

where E_z is the field seen by a particle undergoing acceleration in the cavity. Here we describe the theoretical calculations of the effects of two methods of material layering on P_{loss} in a cylindrical pillbox cavity, while keeping the same acceleration field. The outer (round) wall of the cavity interacts with the fields in a way similar to layered optical coatings [1]. Thus, to greatly reduce the ohmic losses in the round walls we use a set of concentric annuli of alternating high and low dielectric materials, backed by a metal substrate. For the flat walls, the situation is analogous to a shielded coaxial transmission line, for which the layered metal/dielectric structure of [2] has been found to be useful. Such a structure works by effectively increasing the penetration of the fields into the conducting material, and may be understood in terms of an increase in the classical skin depth [3].

II. Cavity Field Description

The electromagnetic fields are taken to satisfy the wave equation, with the assumption of a constant dielectric constant ε and magnetic permeability ($\mu = \mu_0$), and zero net charge density everywhere

$$\nabla^2\psi = \mu_0\varepsilon\,\frac{\partial^2\psi}{\partial t^2} + \mu_0\sigma\,\frac{\partial\psi}{\partial t}\,. \quad (46)$$

The variable ψ is any of the components of $H, E,$ or D.

We will here analyze a TM_{0n0} mode, where n refers to the number of radial nodes of the electric field. The electric field thus points along the axis of the cavity and the magnetic fields are strictly circular.

The geometry of the system is described in Fig. 1. There is layering in both the radial and the longitudinal directions. The geometry is such that the boundaries between layers are all described by cylinders concentric with the z axis or planes of constant z. The flat layers that comprise the cavity wall consist of alternating metal, dielectric,

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